Math Virtual Learning

Calculus AB

Related Rates using Derivatives

May 8, 2020

Calculus AB

Lesson: May 8, 2020

Objective/Learning Target:

Lesson 5 Derivatives Review

Students will complete related rates problems using derivatives.

Warm-Up:

Note: This is a review. For more examples refer back to your notes.

Watch Videos: Related Rates Into

Related Rates Example

Notes:

Related Rates: Problem Solving Strategy

1. **Draw a picture of the physical situation.**

   Don't stare at a blank piece of paper; instead, sketch the situation for yourself. Really.

2. **Write an equation that relates the quantities of interest.**

   A. Be sure to label as a variable any value that changes as the situation progresses; don't substitute a number for it yet.

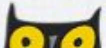
   B. To develop your equation, you will probably use:

   - a simple geometric fact (like for a circle $A = \pi r^2$, or for a cone $V = \frac{1}{3}\pi r^2 h$); or
   - a trigonometric function (like tan θ = opposite/adjacent); or
   - the Pythagorean theorem; or
   - similar triangles.

   Most frequently (> 80% of the time) you will use the **Pythagorean theorem** or **similar triangles**.

3. **Take the derivative with respect to time of both sides of your equation. Remember the Chain Rule.**

4. **Solve for the quantity you're after.**



Example 1:

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\,\text{cm}^3/\text{sec}$ (Figure 4.2). How fast is the radius increasing when the radius is $3\,\text{cm}$?

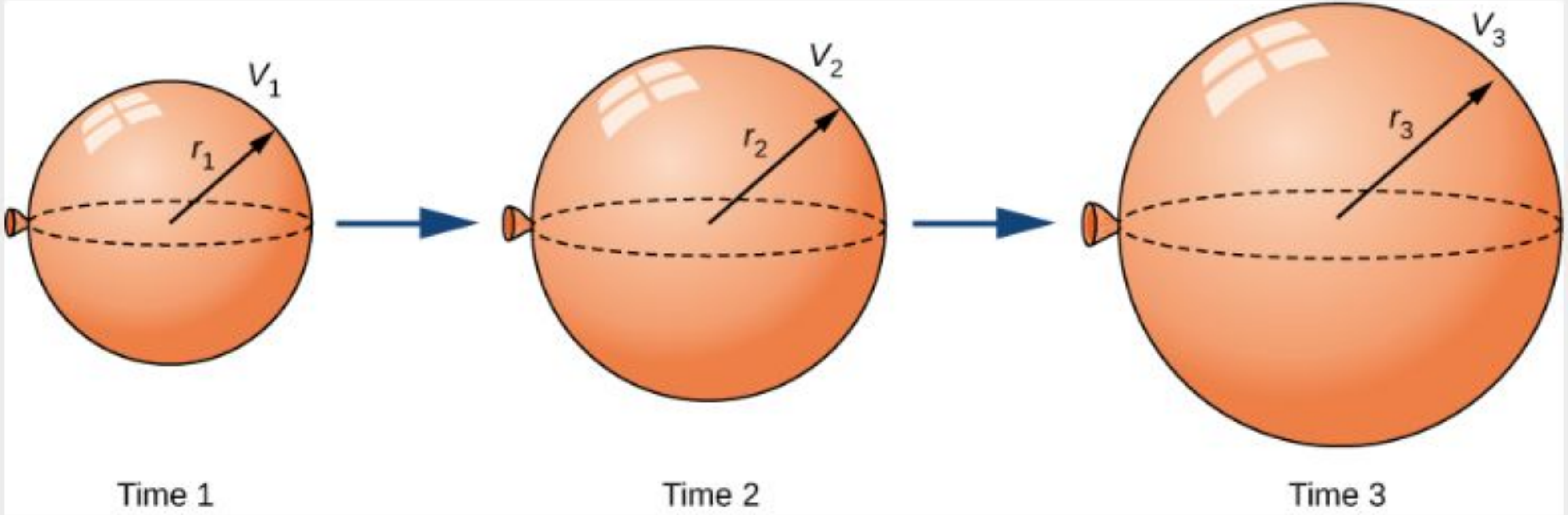


Figure 4.2 As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Example 1 (continued):

Solution

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi [r(t)]^3 \text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi [r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V'(t) = 2\,\text{cm}^3/\text{sec}$. Therefore,

$$2\text{cm}^3/\text{sec} = \left(4\pi [r(t)]^2 \text{cm}^2\right) \cdot \left(r'(t)\,\text{cm/s}\right),$$

Example 1 (continued):

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2} \text{ cm/sec.}$$

When the radius $r = 3\text{ cm}$,

$$r'(t) = \frac{1}{18\pi} \text{ cm/sec.}$$

Example 2:

Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\ \text{ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is $1\ \text{ft}$. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\ \text{ft}$?

Example 2 (continued):

Solution

Step 1: Draw a picture introducing the variables.

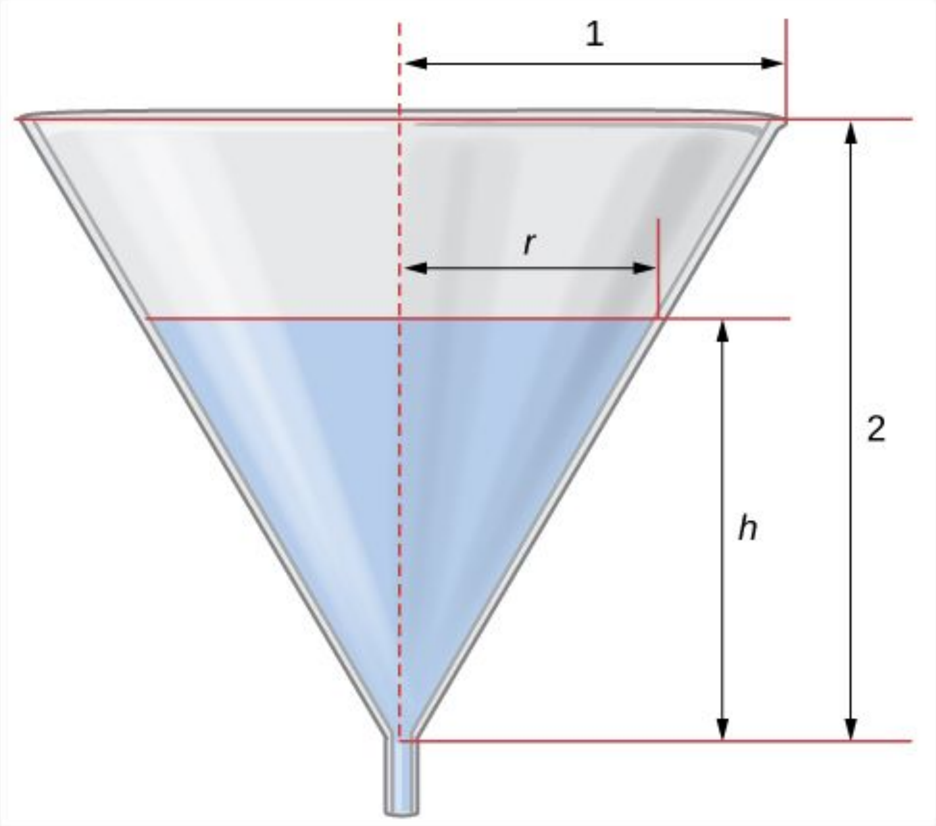


Figure 4.6 Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Example 2 (continued):

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03\,\text{ft/sec}.$

Step 3: The volume of water in the cone is

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Example 2 (continued):

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Therefore,

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 \frac{dh}{dt},$$

which implies

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec}.$$

Practice:

1) A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

2) Oil from an uncapped well is radiating outward in the form of a circular film on the surface of the water. If the radius of the circle is increasing at the rate of 0.5 meters per minute, how fast is the area of the oil film growing at the instant when the radius is 100 m?

Answer Key:

Once you have completed the problems, check your answers here.

1)

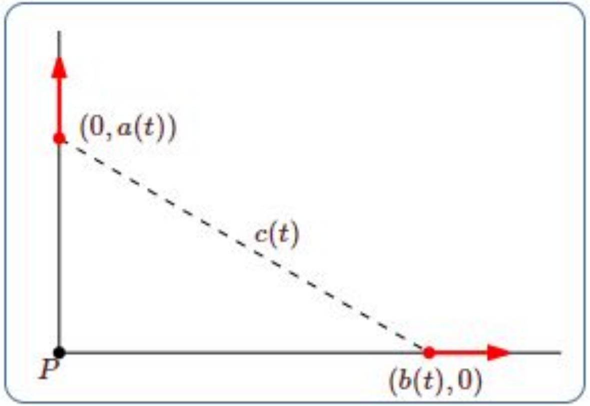


Figure 6.2.4. *Cars moving apart.*

Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$, so

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{a\dot{a} + b\dot{b}}{\sqrt{a^2 + b^2}}.$$

Substituting known values we get:

$$\dot{c} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6\text{km/hr}$$

at the time of interest.

Answer Key:

Once you have completed the problems, check your answers here.

2)

Solution.

Suppose that t is time in minutes, R and A are the radius and area of the circle, respectively.

The rate of change of the area is giveb by the derivative $\frac{dA}{dt}$, where

$$A = \pi R^2.$$

Differentiating the right-hand side of the relation by the chain rule, we get

$$\frac{dA}{dt} = \frac{d}{dt}(\pi R^2) = 2\pi R \frac{dR}{dt}.$$

It is known that $\frac{dR}{dt} = 0.5\,\frac{\text{m}}{\text{min}}$. Therefore, the oil spot is growing at the rate

$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt} = 2\pi R \cdot 0.5 = \pi R.$$

For $R = 100\,\text{m}$, we have

$$\frac{dA}{dt} = 100\pi \approx 314\,\frac{\text{m}^2}{\text{min}}.$$

Additional Practice:

Interactive Practice

More Interactive Practice

Extra Practice with Answers